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THE MOD-2 COHOMOLOGY OF THE BIANCHI GROUPS

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ABSTRACT. The Bianchi groups are a family of discrete subgroups of $PSL_2(\mathbb{C})$ which have group theoretic descriptions as amalgamated products and HNN extensions. Using Bass-Serre theory, we show how the cohomology of these two constructions relates to the cohomology of their pieces. We then apply these results to calculate the mod-2 cohomology ring for various Bianchi groups.

1. INTRODUCTION

The Bianchi groups are the groups $\Gamma_d = PSL_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of algebraic integers of $\mathbb{Q}(\sqrt{-d})$, and d is any positive square-free integer. They can be thought of as generalizations of $PSL_2(\mathbb{Z})$, the modular group. These groups are classical objects, investigated as early as 1892 by Luigi Bianchi, who built fundamental domains for various values of d. Besides their obvious group theoretic interest, the Bianchi groups have been studied from other points of view, some as diverse as 3-manifold theory, the theory of automorphic forms [8], and topological K-theory [4].

In this paper we focus on the calculation of the Bianchi groups' cohomology rings. The difficulty in these calculations lies in that the Bianchi groups are infinite, for $\mathbb{Z} \subset \mathcal{O}_d$ and hence $PSL_2(\mathbb{Z}) \subset PSL_2(\mathcal{O}_d)$. Many of the standard computational techniques of cohomology of groups work best for finite groups, as they depend on knowledge of Sylow p-subgroups or restriction maps to detecting subgroups. Such structures in infinite groups may be too complicated to work with or too difficult to find. The key fact that makes calculation possible for the Bianchi groups is that although these groups are infinite, they are built in stages out of relatively simple subgroups welded together through amalgamations and HNN extensions. We show how amalgamations and HNN extensions affect cohomology in general, then build the cohomology rings of the Bianchi groups in stages.

Some work in this vein has been done separately by Alperin [2], Mendoza [9], and Schwermer and Vogtmann [10]. Alperin calculated the integral homology of $SL_2(\mathcal{O}_3)$ using a simplicial complex based on the group itself. Mendoza constructed a two-dimensional deformation retract of hyperbolic three-space which he used to perform various cohomology calculations with module coefficients. Schwermer and Vogtmann used this Mendoza complex to calculate the integral homology of the five Euclidean Bianchi groups, so named because the rings $\mathcal{O}_d$ for these groups have a Euclidean algorithm. Vogtmann also used the Mendoza complex to find the rational cohomology of all Bianchi groups with $d < 100$ [14].

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Our approach uses group presentations, which provides both the mod-2 cohomology ring and its structure over the Steenrod algebra. This approach better shows how the cohomology classes fit together, as well as the extent to which the cohomology of finite subgroups controls the cohomology of the whole group. It also affords a check on the additive structure determined by Schwermer and Vogtmann in [10].

We do all calculations with coefficients in the field of two elements, $\mathbb{F}_2$, unless otherwise indicated. This is an acceptable reduction for a couple of reasons. First, the calculations simplify considerably. Second, the only possible torsion elements in any Bianchi group have order two or three. From the list of finite subgroups one sees that most of the torsion is in fact of order two. Thus, the choice of $\mathbb{F}_2$ coefficients still yields the majority of the total cohomological information about the Bianchi groups.

This paper is organized as follows: In the next section we develop the machinery we will need for the calculations. In the following three sections we do the actual cohomology calculations to get complete ring structures for the five Euclidean cases and three non-Euclidean ones. We cover the groups Γ_6 and Γ_2 in detail, then give rough outlines for the remaining cases.

The results in this paper are part of my doctoral dissertation completed at the University of Wisconsin, Madison under Alejandro Adem. He initially pointed me towards the Bianchi groups, and I thank him for his many helpful comments and suggestions.

2. Structure Results

The Bianchi groups are less exotic than they might at first appear. As $\mathcal{O}_d$ is a ring of integers, the Bianchi groups are generalizations of $PSL_2(\mathbb{Z})$. For any square-free d, the ring $\mathcal{O}_d$ has a particularly nice algebraic form: as a $\mathbb{Z}$ -module, $\mathcal{O}_d \cong \mathbb{Z} \oplus \mathbb{Z}\omega$, where $\omega = \sqrt{-d}$ for $d \equiv 1,\ 2 \mod 4$ and $\omega = \frac{1+\sqrt{-d}}{2}$ for $d \equiv 3 \mod 4$. Elements of a Bianchi group have a geometric interpretation as well. An element of Γ_d, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with entries in $\mathcal{O}_d$, can be identified with the fractional linear transformation $\frac{Az+B}{Cz+D}$ [3]. In this way g acts by isometries on the Riemann sphere, $\mathbb{C} \cup \infty$, and this action can be uniquely extended to the hyperbolic upper half space $\mathbb{H}^3 = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R}^+ \mid \zeta > 0\}$. The Bianchi groups act properly discontinuously on $\mathbb{H}^3$, so one might hope to use the quotient space for calculations. Unfortunately, the quotient is open and hence not compact, making homology and cohomology calculations difficult. Mendoza overcame this problem by constructing an equivariant deformation retract of $\mathbb{H}^3$ with compact quotient and CW structure [9]. With this complex and the stabilizers of the vertices and edges, one can calculate cohomology rings.

We take a more group theoretic approach, using presentations. The Bianchi groups in this article are built up from infinite cyclic and finite subgroups. This latter family is surprisingly small: one can show that the only possible finite subgroups of a Bianchi group are the cyclic groups of order 2 and 3, denoted $\mathbb{Z}/2$ and $\mathbb{Z}/3$; the elementary abelian group $\mathbf{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$; the symmetric group on three letters $\mathbf{S}_3$; and the alternating group on four letters $\mathbf{A}_4$ [8]. Subsets of this family appear in each of the Bianchi groups considered in this paper, and Γ_1 actually contains them all. Since the cohomology of the Bianchi groups depends on these finite subgroups, it is appropriate that we start with a description of these well-known mod-2 cohomology rings.

Theorem 2.1 ([1]). *The cohomology rings for finite subgroups of Bianchi groups are:*

$$\begin{aligned} H^*(\mathbb{Z}/3) \cong H^*(1) &\cong \mathbb{F}_2, \\ H^*(\mathbf{D}_2) &\cong \mathbb{F}_2[x_1, y_1], \\ H^*(\mathbb{Z}/2) \cong H^*(\mathbf{S}_3) &\cong \mathbb{F}_2[x_1], \\ H^*(\mathbf{A}_4) &\cong \mathbb{F}_2[u_2, v_3, w_3]/(u_2^3 + v_3^2 + w_3^2 + v_3 w_3). \end{aligned}$$

(The subscripts denote the degree of the generators.)

We need the restriction maps in cohomology between these groups and their subgroups. Most are straightforward, but one case, $res^{\mathbf{A}_4}_{\mathbb{Z}/2}$, is a little more involved.

Lemma 2.2. *The map $res^{\mathbf{A}_4}_{\mathbb{Z}/2}$ is given on a suitable choice of generators by:*

$$res^{\mathbf{A}_4}_{\mathbb{Z}/2} u_2 = x_1^2, \qquad res^{\mathbf{A}_4}_{\mathbb{Z}/2} w_3 = x_1^3, \qquad res^{\mathbf{A}_4}_{\mathbb{Z}/2} v_3 = 0.$$

Proof. $\mathbf{D}_2$, the Sylow two subgroup of $\mathbf{A}_4$, fits into the short exact sequence

$$1 \to \mathbf{D}_2 \to \mathbf{A}_4 \to \mathbb{Z}/3 \to 1.$$

As $\mathbf{D}_2$ is normal in $\mathbf{A}_4$ there is an action of $\mathbb{Z}/3$ on $\mathbf{D}_2$ with $H^*(\mathbf{A}_4) = H^*(\mathbf{D}_2)^{\mathbb{Z}/3}$, the invariants under this action. If x and y are the generators of $H^1(\mathbf{D}_2)$, $H^*(\mathbf{A}_4)$ is generated by the following three classes [1]:

$$\begin{aligned} u_2 &= x^2 + xy + y^2, \\ v_3 &= x^2 y + x y^2, \\ w_3 &= x^3 + x^2 y + y^3. \end{aligned}$$

Now choose a copy of $\mathbb{Z}/2$ in $\mathbf{A}_4$. Since all elements of order two are conjugate, we can assume without loss of generality that x is the generator of $H^1(\mathbb{Z}/2)$. Then $res^{\mathbf{A}_4}_{\mathbb{Z}/2} x = x$, $res^{\mathbf{A}_4}_{\mathbb{Z}/2} y = 0$, and the result follows. One can easily calculate the action of the Steenrod squaring operations on the generators of $H^*(\mathbf{A}_4)$ from this description. □

The Bianchi groups are formed in stages from their finite subgroups using two group theoretic constructions, the HNN extension and the amalgamated product. The latter is well known to topologists as well as algebraists, but the HNN extension is a little more unusual. We will use it often in our calculations, and so we review its definition, following the notation in Fine's book [6].

Definition. Let G_1 be a group, G_2 a subgroup, and $\theta : G_2 \to G_1$ a monomorphism. Then an *HNN extension of* G_1 is a group

$$G = \langle t, G_1 \mid t^{-1} g t = \theta(g), g \in G_2 \rangle.$$

G is denoted by $HNN(t, G_1, G_2, G_2)$. G_1 is called the *base* and G_2 is called the *associated subgroup.*

Calculating the cohomology of an HNN extension is not much harder than calculating the cohomology of an amalgamated product if one uses the Bass-Serre theory of *trees* [11]. Summarizing, a tree is a contractible CW complex consisting of zero- and one-cells. Bass and Serre show, for every amalgamated product, how

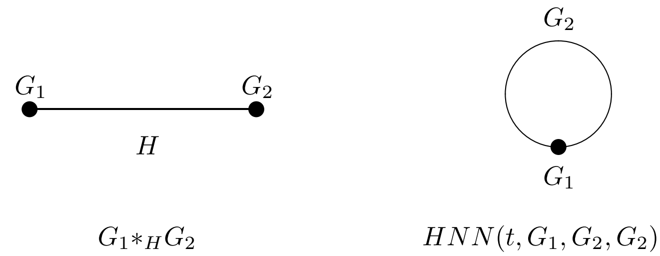

FIGURE 1. Quotient Spaces and Isotropy

to construct a tree and action which has a line segment as a quotient, with the factor groups as the isotropy of the vertices and the amalgamated subgroup as the isotropy of the edge. Similarly, for HNN extensions the analogous construction yields an edge and a single vertex, i.e. a cycle, as the quotient. In this case, the associated subgroup is the isotropy of the edge and the base is the isotropy of the vertex (see Figure 1).

After some homological algebra one can derive the following long exact sequence in cohomology.

Theorem 2.3 ([11])**.** *For G as above, there is a long exact sequence in cohomology*

$$\cdots \xrightarrow{\alpha} \bigoplus_{e\in\Sigma_1} H^{i-1}(G_e) \xrightarrow{\delta} H^i(G) \xrightarrow{\beta} \bigoplus_{v\in\Sigma_0} H^i(G_v) \xrightarrow{\alpha} \bigoplus_{e\in\Sigma_1} H^i(G_e) \xrightarrow{\delta} \cdots .$$

The direct sum is over one edge and two vertices if G is an amalgamated product, and over one edge and one vertex if G is an HNN extension.

Proposition 2.4 ([11])**.** *In the long exact sequence of Theorem 2.3 , the map β is the restriction map. When $G = G_1 *_H G_2$, α is the difference of restriction maps, $\alpha = res^{*}{}^{G_1}_{H} - res^{*}{}^{G_2}_{H}$. When $G = HNN(t, G_1, G_2, G_2)$, the "twisting" induced by θ comes into play, and $\alpha = res^{*}{}^{G_1}_{G_2} - \theta^*$.*

This is enough theory to calculate the cohomology of an amalgamated product or HNN extension as a graded group. In fact, we can recover a good deal of the ring structure as well.

Proposition 2.5. *For G either an HNN extension or an amalgamated product, the kernel of α in Theorem 2.3 is closed under cup products. That is, for $u, v \in H^*(G_v)$, if $\alpha(u) = \alpha(v) = 0$ then $\alpha(u \cup v) = 0$.*

Proof. We do the proof for the case of an HNN extension. If $\alpha(u) = 0$, then $res^*(u) = \theta^*(u)$, and likewise for v. By naturality of the restriction map and the conjugation map,

$$\begin{aligned} res^*(u \cup v) &= res^*(u) \cup res^*(v) \\ &= \theta^*(u) \cup \theta^*(v) \\ &= \theta^*(u \cup v). \end{aligned}$$

□

Remark 2.6. The map β in Theorem 2.3 is a restriction map, so it respects cup products. Thus, as $H^*(G) \xrightarrow{\beta} \bigoplus H^*(G_v) \xrightarrow{\alpha} \bigoplus H^*(G_e)$ is exact, $im(\beta) \cong ker(\alpha)$, and this is an isomorphism of rings.

In the calculations to follow, most classes in $H^*(G)$ have non-trivial images under β, so most of the cohomology ring structure in the Euclidean Bianchi groups comes directly from the cohomology ring structure of the G_v, the finite subgroups. At this point we add some notation. As restriction maps are induced by an inclusion map, i, we use i^* to mean the restriction map on cohomology. We also need one more result, to determine products with classes which arise from the image of δ.

Proposition 2.7 (See 5.6 in [12])**.** *In Theorem 2.3, the map δ respects the cup products in G. Let $i : G_e \to G$ be the inclusion map. Then for $u \in H^p(G)$ and $v \in H^q(G_e)$ we have $\delta(i^*u \cup v) = u \cup \delta(v)$ and $\delta(v \cup i^*u) = \delta(v) \cup u$.*

These sequences and maps can be developed from other points of view. Using the trees associated to amalgamated products and HNN extensions, the equivariant spectral sequence associated to the quotient space collapses at the E_2 page to yield these results. For the amalgamated product, $G = G_1 *_H G_2$, it is also possible to use the commutative diagram of $K(\pi, 1)$'s:

$$\begin{array}{ccc} H & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & G \end{array}$$

where all the maps are injections [5]. This construction yields a Mayer-Vietoris sequence, and the properties follow. This point of view can facilitate some results, for example, an analog to the cohomology of a pointed sum of spaces:

Theorem 2.8 ([5])**.** *The cohomology ring of a free product $G_1 * G_2$ with any coefficients R is the reduced sum of the cohomology rings of G_1 and G_2, namely $H^*(G_1; R)\widetilde{\oplus} H^*(G_2; R)$.*

Corollary 2.9. *As $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$, $H^*(PSL_2(\mathbb{Z})) \cong H^*(\mathbb{Z}/2)$.*

3. The Bianchi Groups Γ_2 and Γ_6

In this section we use results from the previous section to calculate $H^*(\Gamma_2)$ and $H^*(\Gamma_6)$. These are the most interesting (cohomologically) of the cases we consider so we do them in detail; with the exception of Γ_3, the other Bianchi groups pose no new challenges.

3.1. **The Case Γ_2.** Using Flöge's presentation for the group, $\Gamma_2 = \langle A, V, S, M, U; A^2 = S^3 = (AM)^2 = M^2 = V^3 = 1,\ AM = SV^2,\ U^{-1}AU = M,\ U^{-1}SU = V\rangle$ [7]. Set $G = \langle A, V, S, M; A^2 = M^2 = (AM)^2 = S^3 = V^3 = 1,\ AM = SV^2\rangle$ and consider the subgroups

$$G_1 = \langle A, M; A^2 = M^2 = (AM)^2 = 1\rangle \cong \mathbf{D}_2,$$
$$G_2 = \langle S, V; S^3 = V^3 = (SV^2)^2 = 1\rangle \cong \mathbf{A}_4,$$
$$H = \langle AM = SV^2,\ (AM)^2 = 1\rangle \cong \mathbb{Z}/2.$$

Then G is the amalgamated product $G_1 *_H G_2 \cong \mathbf{D}_2 *_{\mathbb{Z}/2} \mathbf{A}_4$ and $\Gamma_2 = HNN(U, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, where $PSL_2(\mathbb{Z}) = \langle A, S\rangle$.

By Theorem 2.3 the cohomology of G, an amalgamated product, can be determined from the long exact sequence

$$(3.1)\qquad 0 \to H^0(G) \xrightarrow{\beta} H^0(\mathbf{D}_2) \oplus H^0(\mathbf{A}_4) \xrightarrow{\alpha} H^0(\mathbb{Z}/2) \xrightarrow{\delta} H^1(G) \xrightarrow{\beta} \cdots.$$

The subgroup $H \cong \mathbb{Z}/2$ is generated by AM, an element of order two which injects into both $\mathbf{A}_4$ and $\mathbf{D}_2$. Let $z_1 \in H^1(\mathbb{Z}/2)$ denote the dual of AM and let x_1 and $y_1 \in H^1(\mathbf{D}_2)$ denote the duals of the group elements A and AM. Say that $H^*(\mathbf{A}_4)$ is generated by u_2, v_3, and w_3. By Lemma 2.2, the map α is defined on these generators by:

$$\alpha(x_1) = \alpha(v_3) = 0, \quad \alpha(y_1^k) = z_1^k, \quad \alpha(u_2^k) = z_1^{2k}, \quad \alpha(w_3^k) = z_1^{3k}.$$

Also, $\alpha(x_1^j y_1^k) = 0$ for all $j > 0$ by the naturality of the cup product. As α is a surjection in all degrees, the long exact sequence (3.1) breaks into short exact sequences

$$0 \to H^k(G) \xrightarrow{\beta} H^k(\mathbf{D}_2) \oplus H^k(\mathbf{A}_4) \xrightarrow{\alpha} H^k(\mathbb{Z}/2) \to 0.$$

This implies $H^*(G) \cong ker(\alpha)$, and by Remark 2.6 this is an isomorphism of rings. We identify five classes which we claim generate $ker(\alpha)$ as a ring: $\bar{x}_1 = (x_1, 0)$, $\bar{y}_2 = (x_1y_1, 0)$, $\bar{u}_2 = (y_1^2, u_2)$, $\bar{w}_3 = (y_1^3, w_3)$, and $\bar{v}_3 = (0, v_3)$. The first four are the minimum required to build the classes $(x_1^j y_1^k, 0)$ in the kernel, and $\bar{v}_3$ is clearly necessary as well. Relations among the five are easy to find, $\bar{x}_1^2\bar{u}_2 = \bar{y}_2^2$, for example, although finding a minimal set takes a little work. This is facilitated by using the Poincaré series, $P(G,t) = \sum_{n\geq 0} dim_{\mathbb{F}_2} H^n(G)t^n$. It is easily determined by α, and can subsequently be used as a check to confirm that all generators and relations have been found. In this case,

$$\begin{aligned} P(G,t) &= \frac{1}{(1-t)^2} + \frac{1+t^3}{(1-t^2)(1-t^3)} - \frac{1}{(1-t)} \\ &= \frac{t^4+3t^3+2t^2+t+1}{(1-t^2)(1-t^3)}. \end{aligned}$$

With some effort we find the relations that agree with the series, which we state in the following lemma. We remove the bars on the classes.

Lemma 3.1. *$H^*(G) \cong \mathbb{F}_2[x_1, y_2, u_2, v_3, w_3]/R$, where R is generated by the set of relations $u_2^3 + v_3^2 + w_3^2 + v_3w_3 = 0$, $x_1^2u_2 = y_2^2$, $x_1w_3 = u_2y_2$, $y_2w_3 = x_1u_2^2$, and $x_1v_3 = y_2v_3 = 0$.*

Most of the Steenrod squaring operations are clear:

	x_1	y_2	u_2	w_3	v_3
Sq^1	x_1^2	$x_1(y_2+u_2)$	v_3	u_2^2	0
Sq^2	0	y_2^2	u_2^2	$u_2(v_3+w_3)$	u_2v_3
Sq^3	0	0	0	w_3^2	v_3^2

To calculate $H^*(\Gamma_2)$ from $H^*(G)$ we use the long exact sequence for an HNN extension from Theorem 2.3:

$$\begin{aligned} 0 &\to H^0(\Gamma_2) \xrightarrow{\beta} H^0(G) \xrightarrow{\alpha} H^0(PSL_2(\mathbb{Z})) \\ &\xrightarrow{\delta} H^1(\Gamma_2) \xrightarrow{\beta} H^1(G) \xrightarrow{\alpha} H^1(PSL_2(\mathbb{Z})) \xrightarrow{\delta} \cdots. \end{aligned}$$

By Proposition 2.4, α is the difference of a restriction map i^* and a twisting map θ^*. Let $w_1 \in H^1(PSL_2(\mathbb{Z}))$ denote the generator dual to $A \in PSL_2(\mathbb{Z})$. The element A is also an element of $\mathbf{D}_2$ which injects into G, so i^* sends $x_1 \in H^1(G)$ to w_1, and all other generators to zero. The twisting map, θ^*, is more interesting. Three torsion is not detected by $\mathbb{F}_2$ coefficients, so we only need consider the twisting component in $\mathbf{D}_2$. Recall that $U^{-1}AU = M = A \cdot AM$. Therefore θ^* sends both x_1 and y_1 to w_1, and hence sends $x_1^j y_1^k$ to w_1^{j+k}. Both θ^* and i^* are zero on $H^*(\mathbf{A}_4)$ so α is too. On $H^*(\mathbf{D}_2)$ we have $\alpha(x_1^j) = 0$, and $\alpha(x_1^j y_1^k) = w_1^{j+k}$ for $k > 0$. Summarizing,

$$\alpha(\bar{x}_1) = \alpha(x_1, 0) = 0, \qquad \alpha(\bar{v}_3) = \alpha(0, v_3) = 0,$$
$$\alpha(\bar{y}_2) = \alpha(x_1 y_1, 0) = w_1^2, \qquad \alpha(\bar{u}_2) = \alpha(y_1^2, u_2) = w_1^2 + 0 = w_1^2,$$
$$\alpha(\bar{w}_3) = \alpha(y_1^3, w_3) = w_1^3 + 0 = w_1^3.$$

Note here that $ker(\alpha)$ is not a module over $H^*(G)$. That is, even though $\alpha(x_1) = 0$, we have that $\alpha(\bar{x}_1\bar{y}_2) = \alpha(x_1^2 y_1, 0) = w_1^3$, **not** 0, as one might expect. One can confirm that the classes $n_1 = \bar{x}_1 = (x_1, 0)$, $m_2 = \bar{y}_2 + \bar{u}_2 = (x_1 y_1 + y_1^2, u_2)$, $m_3 = \bar{x}_1\bar{u}_2 + \bar{w}_3 = (x_1 y_1^2 + y_1^3, w_3)$, and $n_3 = \bar{v}_3 = (0, v_3)$ all lie in the kernel of α. A Poincaré series argument shows that these classes generate most of the kernel as a ring, subject to the relations $n_1 n_3 = m_2^3 + m_3^2 + n_3^2 + m_3 n_3 + n_1 m_2 m_3 = 0$.

In degrees two and higher α is a surjection, so $H^*(\Gamma_2)$ is entirely detected on the kernel of α from this point on, as the long exact sequence breaks into short exact sequences

$$0 \to H^k(\Gamma_2) \xrightarrow{\beta} H^k(G) \xrightarrow{\alpha} H^k(PSL_2(\mathbb{Z})) \to 0.$$

There are a couple of classes in $H^*(\Gamma_2)$ to account for that arise when α is not surjective. In degrees zero and one α factors through zero, yielding $\delta(1) \in H^1(\Gamma_2)$, which we call σ_1 and $\delta(w_1) \in H^2(\Gamma_2)$ which we call σ_2. The class $\delta(1)$ will appear in all HNN extensions, as it arises from the geometry of the group (the map β is always an isomorphism at the zero level). These "HNN classes" are exterior, for α always factors through zero in degree 0 in the long exact sequence for an HNN extension, even with integral coefficients. In this latter case, the resulting class is exterior for dimensional reasons. The class $\delta(1)$ corresponds to this class under the universal coefficient theorem, and so will be exterior as well.

To determine other products, notice that δ factors through zero in degrees two and higher, and that all products with the σ_i must lie in the image of δ by the naturality of β. Thus $\sigma_2^2 = \sigma_1\sigma_2 = 0$, leaving $\sigma_1 n_1 = \sigma_2$ as the only possible product. That this product is non-trivial follows from Proposition 2.7. The class $n_1 = (x_1, 0) \in H^1(\Gamma_2)$ is dual to the group element A. A has dual cohomology classes $x_1 \in H^1(\mathbf{D}_2)$ and $w_1 \in H^1(PSL_2(\mathbb{Z}))$. Thus $i^*(n_1) = w_1$, where $i : PSL_2(\mathbb{Z}) \to \Gamma_2$ is the injection map. Then $\sigma_2 = \delta(w_1) = \delta(1 \cup w_1) = \delta(1 \cup i^*(n_1)) = \delta(1) \cup n_1 = \sigma_1 \cup n_1$.

A calculation of the Poincaré series confirms that we have the complete ring. We keep the classes σ_1 and σ_2 separate from the calculations for readability. We get

$$\begin{aligned} P(\Gamma_2, t) &= \frac{t^4 + 3t^3 + 2t^2 + t + 1}{(1-t^2)(1-t^3)} - \frac{t^2}{(1-t)} + t + t^2 \\ &= \frac{t^6 + t^5 + t^4 + 2t^3 + t^2 + t + 1}{(1-t^2)(1-t^3)} + t + t^2. \end{aligned}$$

Theorem 3.2. *$H^*(\Gamma_2) \cong \mathbb{F}_2[n_1, m_2, n_3, m_3](\sigma_1, \sigma_2)/R$, where R is generated by the set of relations $m_2^3 + m_3^2 + n_3^2 + m_3 n_3 + n_1 m_2 m_3 = 0$, $n_1 n_3 = 0$, and all products of σ_1 and σ_2 with all other classes trivial except for the product $\sigma_1 n_1 = \sigma_2$.*

Most of the Steenrod squares are straightforward to calculate, but we do show one, $Sq^1 m_2$:

$$\begin{aligned} Sq^1 m_2 &= Sq^1(x_1 y_1 + y_1^2, u_2) \\ &= (x_1^2 y_1 + x_1 y_1^2, v_3) = (x_1, 0)(x_1 y_1 + y_1^2, u_2) + (0, v_3) \\ &= n_1 m_2 + n_3. \end{aligned}$$

And the rest:

	σ_1	u_1	m_2	n_3	m_3
Sq^1	0	n_1^2	$n_1 m_2 + n_3$	0	m_2^2
Sq^2	0	0	m_2^2	$m_2 n_3$	$m_2 m_3 + m_2 n_3 + n_1^2 m_3 + n_1 m_2^2$
Sq^3	0	0	0	n_3^2	m_3^2

3.2. **The Case Γ_6.** Flöge's presentation for Γ_6 is $\langle A, B, M, R, S, U, W;\ A^2 = B^2 = M^2 = R^3 = S^3 = (BR)^3 = (BS)^3 = 1,\ AS = MR,\ U^{-1}AU = M,\ U^{-1}SU = R,\ W^{-1}MW = A,\ W^{-1}RBW = SB\rangle$ [7]. Let G be the subgroup generated by the group elements A, B, M, R, S and their relations. The following subgroups of G,

$$\begin{aligned} G_1 &= \langle S, A, B; A^2 = S^3 = B^2 = (BS)^3 = 1\rangle \cong \mathbb{Z}/2 * \mathbf{A}_4, \\ G_2 &= \langle R, B, M; M^2 = R^3 = B^2 = (BR)^3 = 1\rangle \cong \mathbb{Z}/2 * \mathbf{A}_4, \\ H &= \langle B, AS = MR; B^2 = 1\rangle \cong \mathbb{Z}/2 * \mathbb{Z}, \end{aligned}$$

combine to form $G \cong G_1 *_H G_2 \cong (\mathbb{Z}/2 * \mathbf{A}_4) *_{(\mathbb{Z}/2*\mathbb{Z})} (\mathbb{Z}/2 * \mathbf{A}_4)$. The associated subgroups of the HNN extensions, $\langle A,\ S\rangle$ and $\langle M,\ RB\rangle$, are both isomorphic to $PSL_2(\mathbb{Z})$. Set $G_3 = HNN(U, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$. Then $\Gamma_6 = HNN(W, G_3, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$. Thus the cohomology of Γ_6 is calculated in three steps.

Let x_1, y_1, and z_1 denote the duals in cohomology of the group elements A, M and B respectively, and let the copies of $H^*(\mathbf{A}_4)$ in G_1 and G_2 be generated by the classes u_2, v_3, and w_3, and p_2, q_3, and r_3. Finally let $H^1(\mathbb{Z})$ be generated by t_1.

By Theorem 2.3, $H^*(G)$ fits into the long exact sequence

$$0 \to H^0(G) \xrightarrow{\beta} H^0(\mathbf{A}_4 * \mathbb{Z}/2) \oplus H^0(\mathbf{A}_4 * \mathbb{Z}/2) \xrightarrow{\alpha} H^0(\mathbb{Z}/2 * \mathbb{Z}) \xrightarrow{\delta} \cdots .$$

The map α is the difference of restrictions given by Theorem 2.2:

$$\begin{aligned} &\alpha(u_2) = \alpha(p_2) = z_1^2; \qquad \alpha(w_3) = \alpha(r_3) = z_1^3; \\ &\alpha(v_3) = \alpha(q_3) = \alpha(x_1) = \alpha(y_1) = 0. \end{aligned}$$

So α is a surjection in degrees two and higher, with classes $x_1,\ y_1,\ \bar{u}_2 = u_2 + p_2,\ \bar{v}_{31} = v_3,\ \bar{v}_{32} = q_3$, and $\bar{w}_3 = w_3 + r_3$ in the kernel. There are also two classes in degree two that arise as images of δ: $\sigma_2 = \delta(z_1)$ and $\tau_2 = \delta(t_1)$. All products with τ_2 vanish, as this is the only class that originates from a torsion-free subgroup. Products with σ_2 also vanish, as in the case of Γ_2. The relations $\bar{u}_2^3 + \bar{v}_{31}^2 + \bar{v}_{32}^2 + \bar{w}_3^2(\bar{v}_{31} + \bar{v}_{32}) = \bar{v}_{31}\bar{v}_{32} = 0$ are straightforward to find and can be confirmed with a Poincaré series. We remove bars and rename v_{31} as v_3 and v_{32} as $\bar{v}_3$. Summarizing,

Lemma 3.3. *$H^*(G) \cong \mathbb{F}_2[x_1, y_1, u_2, v_3, \bar{v}_3, w_3](\sigma_2, \tau_2)/R$, where R is generated by the relations $u_2^3 + v_3^2 + \bar{v}_3^2 + w_3^2 + w_3(v_3 + \bar{v}_3) = x_1 y_1 = 0$, all products of x_1 and y_1 with $u_2,\ v_3,\ \bar{v}_3$, and w_3 trivial, and all products with σ_2 and τ_2 trivial.*

We add the group element U and calculate the cohomology of the first HNN extension, $H^*(G_3)$. Recall that the associated subgroup, $PSL_2(\mathbb{Z})$, is generated by A and S. We only need consider U's action on A, as S is of order 3. Let u_1 be the element in $H^1(G)$ dual to the group element A, let $i : PSL_2(\mathbb{Z}) \to G_3$ be the injection, and let $U^{-1}AU = M$ induce the twisting. $H^*(G_3)$ fits into the long exact sequence from Theorem 2.3,

$$0 \to H^0(G_3) \xrightarrow{\beta} H^0(G) \xrightarrow{\alpha} H^0(PSL_2(\mathbb{Z})) \xrightarrow{\delta} H^1(G_3) \xrightarrow{\beta} \cdots .$$

The map α is the difference of the restriction map i^* and the twisting map θ^*. As A generates one copy of $\mathbb{Z}/2$ in G, it follows that $i^*(x_1) = \theta^*(y_1) = u_1$, $\alpha(x_1^n) = \alpha(y_1^n) = u_1^n$, and α is a surjection in degrees 1 and higher. The kernel of α is generated by $x_1 + y_1, u_2, v_3, \overline{v}_3$, and w_3. The above long exact sequence breaks into short exact sequences,

$$0 \to H^i(G_3) \xrightarrow{\beta} H^i(G) \xrightarrow{\alpha} H^i(PSL_2(\mathbb{Z})) \to 0.$$

To simplify, let $\bar{x}_1 = x_1 + y_1$ and $\sigma_1 = \delta(1)$.

Lemma 3.4. *$H^*(G_3) \cong \mathbb{F}_2[\bar{x}_1, u_2, v_3, \bar{v}_3, w_3](\sigma_1, \sigma_2, \tau_2)/R$, where R is generated by the relations $u_2^3 + v_3^2 + \bar{v}_3^2 + w_3^2 + w_3(v_3 + \bar{v}_3) = 0$, and all products with $\bar{x}_1,\ \sigma_1,\ \sigma_2$ and τ_2 trivial.*

For $\Gamma_6 \cong HNN(W, G_3, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, let w_1 denote the dual of the group element M in $H^1(PSL_2(\mathbb{Z}))$, where $PSL_2(\mathbb{Z})$ is generated by M and RB. Note that the cohomology classes x_1 and y_1 are dual to A and M, and that neither A nor M is contained in a copy of $\mathbf{A}_4$. As $W^{-1}MW = A$, $i^*(y_1^k) = \theta^*(x_1^k) = w_1^k$. Thus α is zero on $\bar{x}_1$ and all other generators of $H^*(G_3)$, yielding short exact sequences

$$0 \to H^{i-1}(PSL_2(\mathbb{Z})) \xrightarrow{\delta} H^i(\Gamma_6) \xrightarrow{\beta} H^i(G_3) \to 0.$$

This produces the exterior HNN class $\delta(1) = \tau_1 \in H^1(\Gamma_6)$ and $\delta(w_1^k) \in H^{k+1}(\Gamma_6)$ in higher degrees.

These classes generate new products that did not occur in the Bianchi group Γ_2. For $i^*(y_1^k) = w_1^k$ implies that $i^*(\bar{x}_1^k) = w_1^k$. Then by Proposition 2.7 $\delta(w_1^k) = \delta(i^*(\bar{x}_1^k)) = \delta(i^*(\bar{x}_1^k) \cup 1) = \bar{x}_1^k \cup \delta(1) = \bar{x}_1^k \cup \tau_1$. Proposition 2.7 also implies that all other products with τ_1 are zero, as no other class is in the image of δ.

Theorem 3.5. *$H^*(\Gamma_6) \cong \mathbb{F}_2[\bar{x}_1, u_2, v_3, \bar{v}_3, w_3](\sigma_1, \tau_1, \sigma_2, \tau_2)/R$, where R is generated by the relations $u_2^3 + v_3^2 + \bar{v}_3^2 + w_3^2 + w_3(v_3 + \bar{v}_3) = 0$, all products with $\sigma_1,\ \sigma_2$ and τ_2 trivial, and all products with $\bar{x}_1$ and τ_1 trivial except for $\bar{x}_1^n \tau_1$. Moreover,*

$$P(\Gamma_6, t) = \frac{-2t^5 - t^4 + 3t^3 + 2t^2 + t + 1}{(1 - t^2)(1 - t^3)} + 2t + 2t^2.$$

The Steenrod operations for $H^*(\Gamma_6)$ are:

	$\bar{x}_1$	σ_1	σ_2	τ_1	τ_2	u_2	v_3	$\bar{v}_3$	w_3
Sq^1	$\bar{x}_1^2$	0	0	0	0	$v_3 + \bar{v}_3$	0	0	u_2^2
Sq^2	0	0	0	0	0	u_2^2	$u_2 v_3$	$u_2 \bar{v}_3$	$u_2(v_3 + \bar{v}_3 + w_3)$
Sq^3	0	0	0	0	0	0	v_3^2	$\bar{v}_3^2$	w_3^2

4. The Bianchi Groups Γ_5, Γ_{10}, Γ_1, Γ_{11} and Γ_7

These Bianchi groups use the same techniques as the Bianchi groups in the previous section. The calculations for the first two groups, non-Euclidean cases, are more complicated, so we include more details for the determination of their cohomology rings. The last three groups are comparatively simple; we only give a quick summary of the calculations.

4.1. **The Case Γ_5.** The presentation for Γ_5 is $\langle A, B, M, R, S, U, W;\ A^2 = B^2 = M^2 = R^3 = S^3 = (AB)^2 = (BM)^2 = 1,\ AS = MR,\ U^{-1}AU = M,\ U^{-1}SU = R,\ W^{-1}MBW = AB,\ W^{-1}RW = S\rangle$ [7]. Γ_5, like Γ_6, is a double HNN extension with base $G = \langle A, B, M, R, S;\ A^2 = B^2 = M^2 = R^3 = S^3 = (AB)^2 = (BM)^2 = 1,\ AS = MR\rangle$. Let

$$G_1 = \langle S, A, B; S^3 = A^2 = B^2 = (AB)^2 = 1\rangle \cong \mathbb{Z}/3 * \mathbf{D}_2,$$
$$G_2 = \langle R, B, M; R^3 = B^2 = M^2 = (MB)^2 = 1\rangle \cong \mathbb{Z}/3 * \mathbf{D}_2,$$
$$H = \langle B, AS = MR; B^2 = 1\rangle \cong \mathbb{Z}/2 * \mathbb{Z}.$$

Let G be the amalgamated product $G_1 *_H G_2 \cong (\mathbb{Z}/3 * \mathbf{D}_2) *_{(\mathbb{Z}/2*\mathbb{Z})} (\mathbf{D}_2 * \mathbb{Z}/3)$. The twisted subgroups, $\langle A, S\rangle$ and $\langle MB, R\rangle$, are both isomorphic to $PSL_2(\mathbb{Z})$. Set $G_3 = HNN(U, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$ and $\Gamma_5 = HNN(W, G_3, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$.

As $H^*(\mathbb{Z}/3) \cong \mathbb{F}_2$, we refer to both G_1 and G_2 as $\mathbf{D}_2$ for brevity. Let x_1, y_1, s_1, and t_1 denote the duals in cohomology of the group elements A, AB, MB, and M respectively; let z_1 denote the dual to B in $H^1(H)$; and let c_1 generate $H^1(\mathbb{Z})$. Then $H^*(G)$ fits into the long exact sequence

$$0 \to H^0(G) \xrightarrow{\beta} H^0(\mathbf{D}_2) \oplus H^0(\mathbf{D}_2) \xrightarrow{\alpha} H^0(\mathbb{Z}/2 * \mathbb{Z}) \xrightarrow{\delta} H^1(G) \xrightarrow{\beta} \cdots .$$

Here α is the difference of restriction maps induced by the two inclusions $i_1 : H \to G_1$ and $i_2 : H \to G_2$. As $B = A \cdot AB = M \cdot MB$, it follows that $i_1^*(x_1) = i_1^*(y_1) = i_2^*(s_1) = i_2^*(t_1) = z_1$, and $\alpha(x_1^i y_1^j) = \alpha(s_1^i t_1^j) = z_1^{i+j}$. The map α is surjective except in degree one; the three classes $p_1 = (x_1 + y_1, 0)$, $q_1 = (0, s_1 + t_1)$, and $r_1 = (x_1, t_1)$ generate the kernel of α as a ring up to nilpotent elements with one relation, $p_1 q_1 = 0$. We obtain an additional class, $\sigma_2 \in H^2(G)$. This is the image of c_1 under δ, which is not in the image of α as it is the only cohomology class which does not originate from a finite subgroup. As in the case Γ_2, all products with this class are trivial.

Lemma 4.1. *$H^*(G) \cong \mathbb{F}_2[p_1, q_1, r_1](\sigma_2)/R$, where R is generated by the relations $p_1 q_1 = 0$ and all products with σ_2 trivial.*

For $G_3 \cong HNN(U, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, denote by u_1 the element in $H^1(PSL_2(\mathbb{Z}))$ dual to the group element A. We use the long exact sequence associated to the HNN extension

$$0 \to H^0(G_3) \xrightarrow{\beta} H^0(G) \xrightarrow{\alpha} H^0(PSL_2(\mathbb{Z})) \xrightarrow{\delta} H^1(G_3) \xrightarrow{\beta} \cdots .$$

Let $\sigma_1 = \delta(1) \in H^1(G_3)$ be the exterior HNN class. From the injection $i : PSL_2(\mathbb{Z}) \to G$, i^* sends x_1, the element dual to A in $H^1(G)$, to u_1. The twisting part of α is induced by $U^{-1}AU = M$, so θ^* sends t_1 to u_1. This completely describes α, but as in the case of Γ_2 we must take some care when we describe how α acts on products of generators of $H^*(G)$. We find that $\alpha(r_1) = \alpha(x_1, t_1) = u_1 + u_1 = 0$, but for products with $i > 0$ we have $\alpha(p_1^i r_1^j) = \alpha(q_1^i r_1^j) = u_1^{i+j}$.

The classes $l_1 = p_1 + q_1 = (x_1 + y_1, s_1 + t_1)$, $m_1 = r_1 = (x_1, t_1)$, and $m_2 = p_1^2 + p_1 r_1 = (y_1^2 + x_1 y_1, 0)$ generate the kernel of α up to nilpotence. They satisfy the relation $m_2^2 + l_1^2 m_2 + l_1 m_1 m_2 = 0$, and the standard argument shows that all products with σ_1 are trivial.

Lemma 4.2. *$H^*(G_3) \cong \mathbb{F}_2[l_1, m_1, m_2](\sigma_1, \sigma_2)/R$, where R consists of the relations $m_2^2 + l_1^2 m_2 + l_1 m_1 m_2 = 0$, and all products with σ_1 and σ_2 trivial.*

By Theorem 2.3, $H^*(\Gamma_5)$ is calculated from

$$0 \to H^0(\Gamma_5) \xrightarrow{\beta} H^0(G_3) \xrightarrow{\alpha} H^0(PSL_2(\mathbb{Z})) \xrightarrow{\delta} H^1(\Gamma_5) \xrightarrow{\beta} \cdots .$$

Let $\tau_1 = \delta(1) \in H^1(\Gamma_5)$ be the exterior HNN class. Also let $w_1 \in H^1(PSL_2(\mathbb{Z}))$ denote the dual of the element MB and recall that W sends MB to AB, and R to S. We only need consider the first two elements, as the latter two are of order three. From the inclusion $i : PSL_2(\mathbb{Z}) \to G_3$ and the twisting of W, we have that $i^*(s_1) = \theta^*(y_1) = w_1$. Thus $\alpha(s_1) = \alpha(y_1) = w_1$, and $\alpha(t_1) = \alpha(x_1) = 0$. In particular, $\alpha(m_1) = \alpha(x_1, t_1) = 0$, and the image under α of any product in $H^*(G_3)$ containing m_1 will also be zero, as its first component will be divisible by x_1 and its second by t_1. For other products,

$$\alpha(l_1^i) = \alpha((x_1 + y_1)^i, (s_1 + t_1)^i) = w_1^i + w_1^i = 0,$$
$$\alpha(l_1^i m_2^j) = \alpha((x_1 + y_1)^i (y_1^2 + x_1 y_1)^j, 0) = w_1^{i+2j}.$$

The map α is a surjection except in degree one, which yields the class $\tau_2 = \delta(w_1) \in H^2(\Gamma_5)$. The three classes l_1, m_1, and $m_3 = m_1 m_2 = (x_1^2 y_1 + x_1 y_1^2, 0)$ are in the kernel of α; they satisfy the relation $m_3(m_3 + l_1^2 m_1 + l_1 m_1^2) = 0$. By dimensional arguments similar to the case Γ_2, all products with τ_1 and τ_2 are zero.

Theorem 4.3. *$H^*(\Gamma_5) \cong \mathbb{F}_2[l_1, m_1, m_3](\sigma_1, \sigma_2, \tau_1, \tau_2)/R$, where R is generated by the relations $m_3(m_3 + l_1^2 m_1 + l_1 m_1^2) = 0$, and all products of exterior classes with other classes trivial. Moreover,*

$$P(\Gamma_5, t) = \frac{1 + t^3}{(1 - t)^2} + 2t + 2t^2.$$

The Steenrod operations:

	l_1	m_1	m_3	σ_1	σ_2	τ_1	τ_2
Sq^1	l_1^2	m_1^2	0	0	0	0	0
Sq^2	0	0	$m_3(l_1^2 + l_1 m_1 + m_1^2)$	0	0	0	0
Sq^3	0	0	m_3^2	0	0	0	0

4.2. **The Case Γ_{10}.** In many regards this is the most complicated of the groups we have considered so far, as it is a triple HNN extension. However, no new techniques are required for the calculations. As in the other cases, we build up this group in stages. Flöge gives the presentation of Γ_{10} as $\langle A, B, L, S, D, U, W;\ A^2 = B^2 = L^2 = S^3 = (AB)^2 = (AL)^2 = 1\rangle$ with other relations involving D, U, and W that we give in their respective extensions [7]. The base group G_0 has the presentation $\langle A, B, L, S;\ A^2 = B^2 = L^2 = S^3 = (AB)^2 = (AL)^2 = 1\rangle$, which we break into pieces:

$$\begin{aligned} G_{01} &= \langle A, B; A^2 = (AB)^2 = B^2 = 1\rangle \cong \mathbf{D}_2,\\ G_{02} &= \langle A, L; A^2 = (AL)^2 = L^2 = 1\rangle \cong \mathbf{D}_2,\\ G_{03} &= \langle S\rangle \cong \mathbb{Z}/3,\\ H &= \langle A\rangle \cong \mathbb{Z}/2. \end{aligned}$$

With this decomposition,

$$G_0 \cong (G_{01} *_H G_{02}) * G_{03} \cong (\mathbf{D}_2 *_{\mathbb{Z}/2} \mathbf{D}_2) * \mathbb{Z}/3.$$

The first HNN extension, G_1, adds the element D. Its presentation is $\langle G_0, D; D^{-1}ALSD = S^{-1}AB\rangle$, with $\langle ALS\rangle \cong \mathbb{Z}$. G_2, the second HNN extension, adds the group element U. In particular,

$$G_2 = \langle G_1, U; U^{-1}DABD^{-1}U = D^{-1}ALD, U^{-1}LDS^{-1}D^{-1}U = BD^{-1}S^{-1}D\rangle.$$

The subgroup $\langle DABD^{-1}, LDS^{-1}D^{-1}\rangle$ is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}$. The final HNN extension is

$$\Gamma_{10} = \langle G_2, W; W^{-1}BW = U^{-1}LU, W^{-1}D^{-1}SDW = U^{-1}DSD^{-1}U\rangle,$$

where $\langle B, D^{-1}SD\rangle$ is isomorphic to $PSL_2(\mathbb{Z})$.

The calculations for $H^*(G_0)$ are almost identical to those in the first stage of the calculation of $H^*(\Gamma_5)$, as G_{03} is not detected with $\mathbb{F}_2$ coefficients. Let s_1 and t_1 be the duals in cohomology of the group elements A and B in G_{01}, let x_1 and y_1 be the duals of the group elements A and L in G_{02}, and let z_1 be the dual of the group element A in H. In the long exact sequence of Theorem 2.3, α is a surjection with kernel generated by $\bar{x}_1 = (x_1, s_1)$, $\bar{y}_1 = (y_1, 0)$, and $\bar{t}_1 = (0, t_1)$. These classes satisfy the relation $\bar{y}_1\bar{t}_1 = 0$.

Now G_1 is an HNN extension of G_0 where the element D twists a subgroup isomorphic to $\mathbb{Z}$. From the long exact sequence

$$0 \to H^0(G_1) \xrightarrow{\beta} H^0(G_0) \xrightarrow{\alpha} H^0(\mathbb{Z}) \xrightarrow{\delta} H^0(G_1) \xrightarrow{\beta} \cdots$$

we see that the cohomology of G_1 is essentially the same as the cohomology of G_0, except for two new classes in $H^*(G_1)$: the HNN class $\sigma_1 = \delta(1)$, and $\sigma_2 = \delta(c_1)$ where $c_1 \in H^1(\mathbb{Z})$. All non-trivial products with these classes must lie in the image of δ by naturality, so the only possible relation is $\sigma_1^2 = \sigma_2$. But σ_1 is the HNN class, so $\sigma_1^2 = 0$.

Lemma 4.4. *$H^*(G_1) \cong \mathbb{F}_2[\bar{x}_1, \bar{t}_1, \bar{y}_1](\sigma_1, \sigma_2)/R$, where R is generated by the relations $\bar{y}\bar{t} = 0$, and all products with σ_1 and σ_2 zero.*

The calculations for $H^*(G_2)$ are more involved. In the associated subgroup, $\langle DABD^{-1}, LDS^{-1}D^{-1}\rangle \cong \mathbb{Z}/2 * \mathbb{Z}$, we let e_1 denote the class in $H^1(\mathbb{Z})$, and let c_1 be the polynomial generator in $H^1(\mathbb{Z}/2)$. Since conjugation induces the identity map in cohomology, AB and $DABD^{-1}$ represent the same cohomology class. In particular, without loss of generality, c_1 is dual to either AB or $DABD^{-1}$. A similar argument applies to the two group elements AL and $D^{-1}ALD$. Consider the long exact sequence

$$0 \to H^0(G_2) \xrightarrow{\beta} H^0(G_1) \xrightarrow{\alpha} H^0(\mathbb{Z}/2 * \mathbb{Z}) \xrightarrow{\delta} H^1(G_2) \xrightarrow{\beta} \cdots .$$

From the injection $i : \mathbb{Z}/2*\mathbb{Z} \to G_1$, i^* sends both s_1 and t_1 to c_1, as $AB = A \cdot B$; thus $i^*(\bar{x}_1) = i^*(\bar{t}_1) = c_1$. Similarly, the twisting sends both x_1 and y_1 to c_1, so $\theta^*(\bar{x}_1) = \theta^*(\bar{y}_1) = c_1$. It is then easy to verify that $\alpha(\bar{x}_1^j) = 0$ and $\alpha(\bar{x}_1^j \bar{t}_1^k) = \alpha(\bar{x}_1^j \bar{y}_1^k) = c_1^{j+k},\ k > 0$.

The classes $m_1 = \bar{x}_1 = (x_1, s_1)$, $l_1 = \bar{y}_1 + \bar{t}_1 = (y_1, t_1)$, and $m_2 = \bar{x}_1\bar{y}_1 + \bar{y}_1^2 = (x_1y_1 + y_1^2, 0)$ generate most of the kernel of α. As in the case Γ_5, these classes satisfy the relation $m_2^2 + l_1^2 m_2 + l_1 m_1 m_2 = 0$. Two other classes come from the boundary operator, $\tau_1 = \delta(1) \in H^1(G_2)$ and $\tau_2 = \delta(e_1) \in H^2(G_2)$. By similar arguments as before, products with these classes are trivial.

Lemma 4.5. *$H^*(G_2) \cong \mathbb{F}_2[l_1, m_1, m_2](\sigma_1, \sigma_2, \tau_1, \tau_2)/R$, where R is generated by the relations $m_2^2 + l_1^2 m_2 + l_1 m_1 m_2 = 0$, and all products with the four exterior classes trivial.*

Γ_{10} is the final HNN extension. Recall that the associated subgroup is

$$\langle B, D^{-1}SD \rangle \cong PSL_2(\mathbb{Z}).$$

The twisting, θ, in the HNN extension sends B to $U^{-1}LU$, which represents the same cohomology class as L. This situation is identical to the calculations in the final stage of Γ_5. We refer the reader there, and state the result.

Theorem 4.6. *$H^*(\Gamma_{10}) \cong \mathbb{F}_2[l_1, m_1, m_3](\sigma_1, \sigma_2, \tau_1, \tau_2, \eta_1, \eta_2)/R$, where R is generated by the relations $m_3(m_3 + l_1^2 m_1 + l_1 m_1^2) = 0$, and all products of exterior classes with other classes trivial. Moreover,*

$$P(\Gamma_{10}, t) = \frac{1 + t^3}{(1-t)^2} + 3t + 3t^2.$$

Not only does this ring closely match $H^*(\Gamma_5)$, but the Steenrod squares are identical.

4.3. **The Case Γ_1.** The group presentation for Γ_1 is $\langle A, B, C, D; A^3 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BD)^2 = (BC)^2 = 1 \rangle$ ([6], §4.4). Consider the subgroups

$$G_{11} = \langle A, C; A^3 = C^3 = (AC)^2 = 1 \rangle \cong \mathbf{A}_4,$$
$$G_{12} = \langle A, D; A^3 = D^2 = (AD)^2 = 1 \rangle \cong \mathbf{S}_3,$$
$$G_{21} = \langle B, C; B^2 = C^3 = (BC)^2 = 1 \rangle \cong \mathbf{S}_3,$$
$$G_{22} = \langle B, D; B^2 = D^2 = (BD)^2 = 1 \rangle \cong \mathbf{D}_2.$$

The Bianchi group Γ_1 is the amalgamated product $G_1 *_H G_2$, where G_1 and G_2 are themselves amalgamated products and H is the modular group:

$$G_1 = G_{11} *_{\langle A \rangle} G_{12}, \qquad \langle A \rangle \cong \mathbb{Z}/3,$$
$$G_2 = G_{21} *_{\langle B \rangle} G_{22}, \qquad \langle B \rangle \cong \mathbb{Z}/2,$$
$$H = \langle C, D \rangle \cong \mathbb{Z}/2 * \mathbb{Z}/3 \cong PSL_2(\mathbb{Z}).$$

It is elementary to verify that if the amalgamated subgroup is cohomologically trivial, then Theorem 2.8 holds and the cohomology of the group is the reduced sum of the cohomology of the factors. In our case, $\mathbb{Z}/3$ has order prime to 2, so $H^i(G_1) \cong H^i(\mathbf{A}_4)\widetilde{\oplus}H^i(\mathbf{S}_3)$. In another straightforward calculation, $H^*(G_2) \cong H^*(\mathbf{D}_2)$. For the final amalgamation, one finds that $H^*(\Gamma_1) \cong H^*(\mathbf{A}_4)\widetilde{\oplus}H^*(\mathbf{D}_2)$. This isomorphism also extends to cup products and Steenrod squares.

4.4. **The Case Γ_{11}.** We use Fine's presentation, $\Gamma_{11} = \langle A, T, U; (U^{-1}AUAT)^3 = A^2 = (AT)^3 = [T, U] = 1\rangle$ ([6], §4.3). Set $S = AT$, $W = AU$, $M = W^{-1}AW$, and $V = W^{-1}SW$. Replace the relation $(U^{-1}AUAT)^3 = 1$ with the relation $(ATU^{-1}AU)^3 = (SM)^3 = 1$. Note that

$$\begin{aligned} AV &= AW^{-1}SW = AU^{-1}ASW = ATT^{-1}U^{-1}AATW \\ &= ST^{-1}U^{-1}TW = SU^{-1}W = SW^{-1}AW = SM. \end{aligned}$$

The new presentation for Γ_{11} is $\langle A, V, S, M, W; A^2 = S^3 = (AV)^3 = V^3 = M^2 = 1,\ AV = SM,\ W^{-1}AW = M,\ W^{-1}SW = V\rangle$. Set

$$\begin{aligned} G &= \langle A, V, S, M; A^2 = S^3 = (AV)^3 = V^3 = M^2 = 1,\ AV = SM\rangle, \\ G_1 &= \langle A, V; A^2 = V^3 = (AV)^3 = 1\rangle \cong \mathbf{A}_4, \\ G_2 &= \langle S, M; M^2 = S^3 = (SM)^3 = 1\rangle \cong \mathbf{A}_4, \\ H &= \langle AV = SM,\ (AV)^3 = 1\rangle \cong \mathbb{Z}/3. \end{aligned}$$

We can write G as the amalgamated product $G_1 *_H G_2 \cong \mathbf{A}_4 *_{\mathbb{Z}/3} \mathbf{A}_4$. Then $\Gamma_{11} = HNN(W, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, with $PSL_2(\mathbb{Z}) = \langle A, S\rangle$.

The cohomology of the base group G is directly calculated using Theorem 2.8, as $\mathbb{Z}/3$ has trivial cohomology in $\mathbb{F}_2$ coefficients. This yields $H^*(G) \cong \mathbb{F}_2[u_2, q_2, v_3, r_3, w_3, s_3]/R$, where R is the set of relations $u_2^3 + v_3^2 + w_3^2 + v_3w_3 = 0$, $q_2^3 + r_3^2 + s_3^2 + r_3s_3 = 0$, and all products of u_2, v_3, and w_3 with q_2, r_3, and s_3 trivial.

To find Γ_{11}, we fit this information into the long exact sequence

$$0 \to H^0(\Gamma_{11}) \xrightarrow{\beta} H^0(G) \xrightarrow{\alpha} H^0(PSL_2(\mathbb{Z})) \xrightarrow{\delta} \cdots.$$

The map α is the difference of restriction and twisting maps. Notice that the group element A is an element of order two in both $PSL_2(\mathbb{Z})$ and G_1, and that W's twisting, $W^{-1}AW = M$, sends an element of $PSL_2(\mathbb{Z})$ to an element of G_2. Let w_1 denote A's dual in $H^1(PSL_2(\mathbb{Z}))$. Then

$$i^*(u_2) = \theta^*(q_2) = w_1^2, \quad i^*(v_3) = \theta^*(r_3) = 0, \quad i^*(w_3) = \theta^*(s_3) = w_1^3.$$

The map α is a surjection in degrees two and higher, with classes $\bar{u}_2 = (u_2, q_2)$, $\bar{v}_{31} = (v_3, 0)$, $\bar{v}_{32} = (0, r_3)$, and $\bar{w}_3 = (w_3, s_3)$ in the kernel. We get two other classes, the HNN class $\delta(1) \in H^1(\Gamma_{11})$, and $\delta(w_1) \in H^2(\Gamma_{11})$ which we call σ_1 and σ_2. By similar arguments as in the case Γ_2, all products with these classes are zero. The relations $\bar{u}_2^3 + \bar{v}_{31}^2 + \bar{v}_{32}^2 + \bar{w}_3^2(\bar{v}_{31} + \bar{v}_{32}) = \bar{v}_{31}\bar{v}_{32} = 0$ are easy to find, and can be confirmed with a Poincaré series.

The reader may wish to compare these calculations with the case Γ_6, which has similar classes arising from an amalgamated product. For readability, we remove the bars from all classes, rename v_{31} by v_3, and rename v_{32} by $\bar{v}_3$. We do not include the Steenrod squares, as these are identical to the case Γ_6.

Theorem 4.7. *$H^*(\Gamma_{11}) \cong \mathbb{F}_2[u_2,\ v_3,\ \bar{v}_3,\ w_3](\sigma_1,\ \sigma_2)/R$, where R is generated by the set of relations $u_2^3 + w_3^2 + v_3^2 + \bar{v}_3^2 + w_3(v_3 + \bar{v}_3) = 0$, $v_3\bar{v}_3 = 0$, and all products* *with σ_1 and σ_2 trivial. Moreover,*

$$P(\Gamma_{11}, t) = \frac{t^6 + 2t^3 + 1}{(1-t^2)(1-t^3)} + t + t^2.$$

4.5. **The Case Γ_7.** Fine's presentation for Γ_7 is $\langle A, T, U; (U^{-1}AUAT)^2 = A^2 = (AT)^3 = [T, U] = 1\rangle$ ([6], §4.3). This differs from Γ_{11}'s presentation by only the exponent of $U^{-1}AUAT$; analogous transformations result in the following presentation for the group: $\langle A, V, S, M, W; A^2 = S^3 = (AV)^2 = V^3 = M^2 = 1,\ AV = SM,\ W^{-1}AW = M,\ W^{-1}SW = V\rangle$. Set

$$\begin{aligned}
G &= \langle A, V, S, M; A^2 = S^3 = (AV)^2 = V^3 = M^2 = 1,\ AV = SM\rangle,\\
G_1 &= \langle A, V; A^2 = V^3 = 1\rangle \cong \mathbf{S}_3,\\
G_2 &= \langle S, M; M^2 = S^3 = 1\rangle \cong \mathbf{S}_3,\\
H &= \langle AV = SM,\ (AV)^2 = 1\rangle \cong \mathbb{Z}/2.
\end{aligned}$$

As before, $G \cong \mathbf{S}_3 *_{\mathbb{Z}/2} \mathbf{S}_3$ and $\Gamma_7 = HNN(W, G, PSL_2(\mathbb{Z}), PSL_2(\mathbb{Z}))$, where $PSL_2(\mathbb{Z}) = \langle A, S\rangle$.

This case is similar to, but easier than, the case Γ_{11}. Briefly, one finds that $H^*(G) \cong \mathbb{Z}/2$, and that the map $i^* : H^*(\Gamma_7) \to H^*(PSL_2(\mathbb{Z}))$ is non-zero.

Theorem 4.8. *$H^*(\Gamma_7) \cong \mathbb{F}_2[y_1](x_1)$, and $P(\Gamma_7, t) = \dfrac{1+t}{1-t}$.*

The Steenrod squares on the two generators are:

	x_1	y_1
Sq^1	0	y_1^2

5. The Case Γ_3.

We use Flöge's presentation for the group, $\Gamma_3 = \langle A, L, K, S, M;\ A^2 = L^3 = (AL)^2 = K^3 = S^3 = (KS)^2 = M^2 = (MS)^3 = 1,\ AL = K^2S^2,\ AL^2 = M\rangle$ [7]. We break this into subgroups,

$$\begin{aligned}
G_1 &= \langle A, L; A^2 = L^3 = (AL)^2 = 1\rangle \cong \mathbf{S}_3,\\
G_2 &= \langle K, S; K^3 = S^3 = (K^2S^2)^2 = 1\rangle \cong \mathbf{A}_4,\\
G_3 &= \langle M, S; M^2 = S^3 = (MS)^3 = 1\rangle \cong \mathbf{A}_4,\\
H_1 &= \langle AL = K^2S^2\rangle \cong \mathbb{Z}/2,\\
H_2 &= \langle S\rangle \cong \mathbb{Z}/3,\\
H_3 &= \langle AL^2 = M\rangle \cong \mathbb{Z}/2.
\end{aligned}$$

These subgroups fit into a "triangular product," which is related to an amalgamated product. Optimistically, as in the other cases, this group would act on a tree with a cyclic quotient and the above groups as isotropy. Unfortunately, we have the following result, due to Serre.

Theorem 5.1 ([11]). *Say Γ_3 acts on a tree X. Then $X^{\Gamma_3} \neq \emptyset$, where X^G represents the fixed points of X under the action of G.*

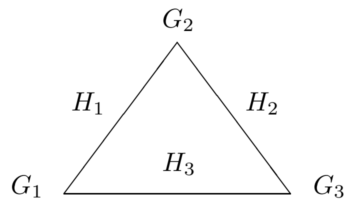

FIGURE 2. Fundamental domain for Γ_3

As a corollary, the quotient graph of any tree on which Γ_3 acts will contain at least one cell which has Γ_3 as an isotropy group. Clearly this quotient will be unusable for cohomology calculations. Fortunately, there is another option: the Mendoza complex mentioned in the first section. In his thesis, Mendoza shows, for any Euclidean Bianchi group Γ_d, how to construct a 2-dimensional deformation retract of $\mathbb{H}^3$ on which Γ_d acts with finite isotropy and compact quotient. He then explicitly calculates this region for the five Euclidean cases. For Γ_3, the quotient is the compact cellular complex with isotropy shown in Figure 2 [9].

The isotropy of the 2-cell is the identity element. To relate the cohomology groups we need a gadget to take the place of the long exact sequence used in the other examples. This is the equivariant spectral sequence. Details are in Brown's book [5], but we quickly summarize them here.

For a group G, say X is a cellular space on which G acts and let $C^*(X)$ be the cellular cochain complex. Say also that F is a projective resolution of $\mathbb{F}_2$ over $\mathbb{F}_2G$. We define $H^*(G, C^*(X))$ to be $H^*(Hom_{\mathbb{F}_2}(F, C^*(X)))$. These are known as the *equivariant cohomology groups of* X, denoted by $H^*_G(X)$. Notice that if X is contractible, as in our case, then $H^*_G(X) \cong H^*(G)$. From analysis of the horizontal and vertical filtrations of the double complex $Hom_{\mathbb{F}_2}(F, C^*(X))$ we get a spectral sequence with $E_1^{pq} = H^q(G, C^p(X))$ which converges to $H^{p+q}(G)$. By Shapiro's lemma, we can also identify E_1^{pq} with $\prod_{\sigma \in \Sigma_p} H^q(G_\sigma)$, where Σ_p is a set of representatives of p-cells in X/G.

This spectral sequence has a number of desirable properties. First, the spectral sequence has a product $E_r^{pq} \otimes E_r^{st} \to E_r^{p+s,q+t}$ which is compatible with the standard cup product on $H^*(G)$. Second, the products in $E_2^{0,*}$, the vertical edge, are compatible with the products in $\prod_{\sigma \in \Sigma_0} H^q(G_\sigma)$. Third, the differential d_1 is a difference of restriction and twisting maps based on inclusion and identification in the quotient complex X/G. When this quotient is a line segment or a single edge and vertex, the spectral sequence consists of only two columns and collapses at the E_2 term to give a Wang sequence that is the long exact sequence we had before. In this case the differential d_1 is the same as the map α.

In the calculation of $H^*(\Gamma_3)$ we have $E_1^{p,q} = 0$ for $p > 1$, with the exception of $E_1^{2,0} \cong \mathbb{F}_2$. Thus in some sense, Γ_3 only "barely requires" the use of this spectral sequence. We write out the E_1 term of the spectral sequence in Figure 3, showing the differentials and cohomology groups in a generic row.

Let $H^*(G_1) \cong \mathbb{F}_2[x_1]$, $H^*(G_2) \cong \mathbb{F}_2[u_2, v_3, w_3]/R$, and $H^*(G_3) \cong \mathbb{F}_2[q_2, r_3, s_3]/R$ (where R are the relations for $H^*(\mathbf{A}_4)$). Likewise, let $H^*(H_1) \cong \mathbb{F}_2[y_1]$, $H^*(H_2) \cong \mathbb{F}_2$, and $H^*(H_3) \cong \mathbb{F}_2[z_1]$. At the zero level, let η_1, η_2, and η_3 generate $H^0(G_i)$, let μ_1, μ_2, and μ_3 generate $H^0(H_i)$, and let ν generate $H^0(\{1\})$. Notice that the elements AL and AL^2 in $\mathbf{S}_3$ are conjugate. Thus they represent the same cohomology

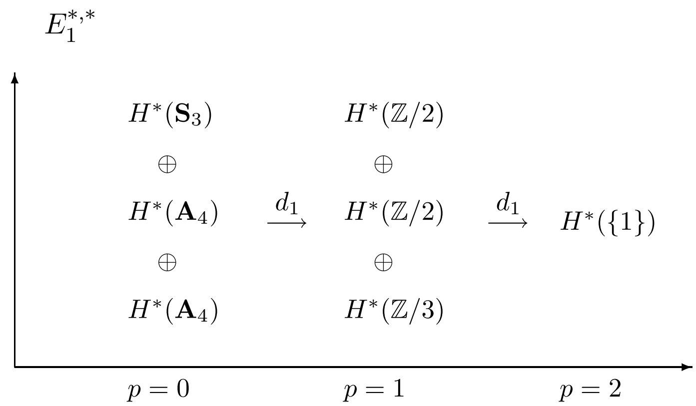

FIGURE 3. The E_1 page

class, x_1, in $H^1(G_1)$. In the previous cases α was the key map in understanding the cohomology, so it is no surprise that here we want to understand the differential d_1. From the quotient diagram in Figure 2 we see that no edges are identified, so d_1 consists only of restriction maps. The inclusion of the isotropy of the edges into the isotropy of the vertices determines d_1's action on the cohomology generators:

$$d_1(\eta_1) = \mu_1 + \mu_3, \qquad d_1(\eta_2) = \mu_1 + \mu_2, \qquad d_1(\eta_3) = \mu_2 + \mu_3,$$
$$d_1(\mu_1) = d_1(\mu_2) = d_1(\mu_3) = \nu,$$
$$d_1(u_2) = y_1^2, \qquad d_1(v_3) = 0, \qquad d_1(w_3) = y_1^3,$$
$$d_1(q_2) = z_1^2, \qquad d_1(r_3) = 0, \qquad d_1(s_3) = z_1^3,$$
$$d_1(x_1) = y_1 + z_1, \qquad d_1(\nu) = 0.$$

So $d_1 : E_1^{0,p} \to E_1^{1,p}$ is a surjection for $p \geq 2$. At $E_1^{0,0}$, $dim_{\mathbb{F}_2} ker(d_1) = 1$ (generated by $\eta_1 + \eta_2 + \eta_3$), and $dim_{\mathbb{F}_2} im(d_1) = 2$. At $E_1^{1,0}$, $dim_{\mathbb{F}_2} ker(d_1) = 2$ (generated by $\mu_1 + \mu_2$ and $\mu_2 + \mu_3$), and $dim_{\mathbb{F}_2} im(d_1) = 1$. At $E_1^{0,2}$, $dim_{\mathbb{F}_2} ker(d_1) = 1$. Thus $dim_{\mathbb{F}_2}(E_2^{0,0}) = 1$, and nothing else from that row survives to E_∞. Also, note that d_1 is not onto at $E_1^{1,1}$.

On the E_2 page, there are no non-zero classes in the second column. Therefore $d_2 = d_\infty = 0$, and $E_2 \cong E_\infty$. From the first row up, all classes in the first column are in the image of d_1, with the exception of one class in $E_1^{1,1}$. This class, which we call σ_2, will survive to E_∞. In the zero column, there are a number of classes which are in the kernel of d_1. In particular (x_1^2, u_2, q_2), (x_1^3, w_3, s_3), $(0, v_3, 0)$, and $(0, 0, r_3)$ and the ideal generated by these classes all survive to the E_∞ page. Call these classes $\bar{u_2}, \bar{w_3}, \bar{v}_{31}$, and $\bar{v}_{32}$ respectively. By the cup product compatibility of this vertical edge, these classes satisfy the relations

$$\bar{u_2}^3 + \bar{w_3}^2 + \bar{v}_{31}^2 + \bar{v}_{32}^2 + \bar{w_3}(\bar{v}_{31} + \bar{v}_{32}) = 0, \quad \bar{v}_{31}\bar{v}_{32} = 0.$$

We also need to determine all products with the class σ_2. This class lies in $E_2^{1,1}$, and, by the compatibility of the spectral sequence with the cup product, any product with this element must lie in the first column or to its right. But σ_2 is the only class in this column, and so it must multiply trivially with all other classes, including itself. We remove the bars for brevity, and as in the case Γ_6 rename the class $\bar{v}_{31}$ as v_3 and the class $\bar{v}_{32}$ as $\bar{v}_3$. The Poincaré series for $H^*(\Gamma_3)$ is almost identical to the case Γ_{11}, as are the final ring structure and the Steenrod squares. We refer the reader to the details of that case, and simply state the result.

Theorem 5.2. *$H^*(\Gamma_3) \cong \mathbb{F}_2[u_2, v_3, \bar{v}_3, w_3](\sigma_2)/R$, where R is generated by the relations $u_2^3 + w_3^2 + v_3^2 + \bar{v}_3^2 + w_3(v_3 + \bar{v}_3) = 0$, $v_3\bar{v}_3 = 0$, and all products with σ_2 trivial. Moreover,*

$$P(\Gamma_3, t) = \frac{t^6 + 2t^3 + 1}{(1 - t^2)(1 - t^3)} + t^2.$$

Remark 5.3. Upon review, there are many similarities between the cohomology rings calculated above. The cases Γ_5 and Γ_{10} are a close match, as are the cases Γ_6, Γ_{11}, and Γ_3, despite their very different group presentations. This is probably a reflection of how the Bianchi groups are fit together from their finite subgroups; there are just so many possible ways to combine them. Of course, a sample of eight examples is a bit sparse. It would be interesting to try some other cases and see how their cohomology rings compare to the known examples. The hard part is building the proper group presentations.

The Poincaré series provides a check on previous calculations of the homology of the Bianchi groups. Using the universal coefficient theorem, one can relate homology with $\mathbb{Z}$ coefficient to our cohomology calculations. This shows that there are a few inaccuracies in [10]. In particular, the results for $H^*(\Gamma_1)$ are short a copy of $\mathbb{Z}/2$ in dimensions 5 and 9 mod 12; and for the case Γ_3 there are two copies too many in dimensions 6 mod 12.

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